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Milnor numbers and Euler obstruction*

José Seade, Mihai Tibăr and Alberto Verjovsky

Abstract. Using a geometric approach, we determine the relations between the local Euler obstruction Eu_f of a holomorphic function f and several generalizations of the Milnor number for functions on singular spaces.

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Mathematical subject classification: 32S05, 32S65, 32S50, 14C17.

1 Introduction

In the case of a nonsingular germ (X, x_0) and a function f with an isolated critical point at x_0, the following three invariants coincide (for (c), up to sign):

(a) the Milnor number of f at x_0, denoted $\mu(f)$;

(b) the number of Morse points in a Morsification of f;

(c) the Poincaré-Hopf index of $\overline{\mathrm{grad}\ f}$ at x_0;

This fact is essentially due to Milnor's work in the late sixties [Mi]. There exist extensions of all these invariants to the case when (X, x_0) is a singular germ, but they do not coincide in general. One of the extensions of (c) is the Euler obstruction of f at x_0, denoted $\mathrm{Eu}_f(X, x_0)$. This was introduced in [BMPS]; roughly, it is the obstruction to extending the conjugate of the gradient of the function f as a section of the Nash bundle of (X, x_0). It measures how far the local Euler obstruction is from satisfying the local Euler condition with respect to f in bivariant theory. It is then natural to compare $\mathrm{Eu}_f(X, x_0)$ to the Milnor

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number of f in the case of a singular germ (X, x_0). This has been also a question raised in [BMPS].

The main idea of this paper is that, for singular X, the Euler obstruction $\mathrm{Eu}_f(X, x_0)$ is most closely related to (b). We use the homological version of the bouquet theorem for the Milnor fiber given in [Ti], which relates the contributions in the bouquet to the number of Morse points. Through this relation, one may compare $\mathrm{Eu}_f(X, x_0)$ to the highest Betti number of the Milnor fiber of f. In case X has *Milnor's property*, the comparison is optimal and yields a general inequality, see §3.1. We further compare $\mathrm{Eu}_f(X, x_0)$ with two different generalizations of the Milnor number for functions with isolated singularity on singular spaces, one due to [Lê3], the other to [Go, MS] for curve singularities and to [IS] for functions on isolated complete intersection germs in general. In case when the germ (X, x_0) is an isolated complete intersection singularity, we use in addition the GSV-index of vector fields [GSV] to completely determine the relations between $\mathrm{Eu}_f(X, x_0)$, the Milnor number of f and the GSV-index attached to f.

2 Euler obstruction and Morsification of functions

Let (X, x_0) denote the germ at some point x_0 of a reduced pure dimensional complex analytic space embedded in $\mathbb{C}^N$, for some N. Consider a Whitney stratification $\mathcal{W}$ of some representative X of (X, x_0).[1] Let W_0 be the stratum containing x_0 and let $W_1, \ldots W_q$ be the finitely many strata of X having x_0 in their closure, other than W_0. Such a Whitney stratification is not unique, but there is a unique "most economical" one (having maximal strata, in the sense of inclusion of strata) by results of Mather and Teissier. We shall not need here this unique stratification, we just fix any Whitney stratification $\mathcal{W}$. Once we have fixed the stratification, its germ at x_0 does not depend on the choice of the representative of X. The next definitions (of general functions, resp. functions with singularity) will depend on the the chosen stratification but not on the local embedding in a nonsingular space.

Let us denote by $F : (\mathbb{C}^N, x_0) \to (\mathbb{C}, 0)$ some extension of f.

Definition 2.1. (Lazzeri '73, Benedetti '77, Pignoni '79, Goresky-MacPherson '83 [GM, p.52]). *One says that* $f : (X, x_0) \to \mathbb{C}$ *is a* general function *if* $\mathrm{d}F_{x_0}$ *does not vanish on any limit of tangent spaces to* W_i, $\forall i \neq 0$, *and to* $W_0 \setminus \{x_0\}$. *One says that* $f : (X, x_0) \to \mathbb{C}$ *is a* stratified Morse function germ *if:*

[1] we shall use throughout the paper the notation X for some representative of the germ (X, x_0).

$\dim W_0 \geq 1$, f *is general with respect to the strata* W_i, $i \neq 0$ *and the restriction* $f_{|W_0} : W_0 \to \mathbb{C}$ *has a Morse point at* x_0.

We have emphasized the special role of the dimension of W_0 by the following reason: if $\dim W_0 = 0$ (which means $W_0 = \{x_0\}$) then general functions on the germ (X, x_0) do exist, whereas Morse functions do not.

Let us recall some definitions and notations from [BMPS]. The complex conjugate of the gradient of the extension F projects to the tangent spaces of the strata of X into a vector field, which may not be continuous. One can make it continuous by "tempering" it in the neighborhoods of "smaller" strata. One gets a well-defined continuous stratified vector field, up to stratified homotopy, which we denote by $\operatorname{grad}_X f$. We shall call it briefly *the gradient vector field*.

Let now $f : (X, x_0) \to \mathbb{C}$ be a function with *isolated singularity at* x_0 with respect to the stratification $\mathcal{W}$. This means by definition that f is a general function at any point $x \neq x_0$ of some representative X of the germ (X, x_0). Then $\operatorname{grad}_X f$ has an isolated zero at x_0. If $\nu : \widetilde{X} \to X$ is the Nash blow-up of X and $\widetilde{T}$ is the Nash bundle over $\widetilde{X}$, then $\operatorname{grad}_X f$ lifts canonically to a never-zero section $\widetilde{\operatorname{grad}_X} f$ of $\widetilde{T}$ restricted to $\widetilde{X} \cap \nu^{-1}(X \cap S_\varepsilon)$, where S_ε is a small enough sphere around x_0, given by Milnor's result [Mi, Cor. 2.8]. Following [BMPS], the obstruction to extend $\widetilde{\operatorname{grad}_X} f$ without zeros throughout $\nu^{-1}(X \cap B_\varepsilon)$ is denoted by $\mathrm{Eu}_f(X, x_0)$ and is called the *local Euler obstruction of* f.

Example 2.2. If the germ (X, x_0) is nonsingular, then the Nash blow-up of the representative X can be identified to X itself, the Nash bundle is the usual tangent bundle of X and $\mathrm{Eu}_f(X, x_0)$ is, by definition, the Poincaré-Hopf index of $\operatorname{grad}_X f$ at x_0. From [Mi, Th.7.2] one deduces: $\mathrm{Eu}_f(X, x_0) = (-1)^{\dim_{\mathbb{C}} X}\mu$, where μ is the Milnor number of f. It is also easy to prove (see [BLS, BMPS]) that if (X, x_0) is any singular space but f is a general function germ at x_0, then the obstruction $\mathrm{Eu}_f(X, x_0)$ is zero.

We claim that a natural way to study Eu_f is to split it according to a Morsification of f. We prove the following general formula for holomorphic germs with isolated singularity:

Proposition 2.3. *Let* $f : (X, x_0) \to (\mathbb{C}, 0)$ *be a holomorphic function with isolated singularity at* x_0. *Then*

$$\mathrm{Eu}_f(X, x_0) = (-1)^{\dim_{\mathbb{C}} X}\alpha_q,$$

where α_q *is the number of Morse points on* $W_q = X_{\mathrm{reg}}$ *in a generic deformation of* f.

Proof. We Morsify the function f, i.e. we consider a small analytic deformation f_λ of f such that f_λ only has stratified Morse points within the ball B and it is general in a small neighborhood of x_0. (See, for instance, the Morsification Theorem 2.2 in [Lê2].)

Since f_λ is a deformation of f, it follows that $\mathrm{grad}_X\, f$ is homotopic to $\mathrm{grad}_X\, f_\lambda$ over the sphere $X \cap \partial B$, so the obstructions to extend their lifts to $\nu^{-1}(X \cap B)$ without zeros are equal.

On the other hand, the obstruction corresponding to $\mathrm{grad}_X\, f_\lambda$ is also equal to the sum of local obstructions due to the Morse points of f_λ. Lemma 4.1 of [STV] shows that the local obstruction at a stratified Morse point is zero if the point lies in a lower dimensional stratum. So the points that only count are the Morse points on the stratum X_{reg} and, at such a point, the obstruction is $(-1)^{\dim_{\mathbb{C}} X}$, as explain above in Example 2.2. $\square$

Remark 2.4. The Euler obstruction is defined via the Nash blow-up and the latter only takes into account the closure of the tangent bundle over the regular part X_{reg}. Since the other strata are not counting in the Nash blow-up, it is natural that they do not count for $\mathrm{Eu}_f(X, x_0)$ neither. The number α_q does not depend on the chosen Morsification, by a trivial connectedness argument. We refer to [STV] for more about α_q and other invariants of this type, which enter in a formula for the *global Euler obstruction* of an affine variety $Y \subset \mathbb{C}^N$.

Remark 2.5. The number α_q may be interpreted as the intersection number within $T^*\mathbb{C}^N$ between $\mathrm{d}F$ and the conormal $T^*_{X_{\mathrm{reg}}}$. Therefore our Proposition 2.3 may be compared to [BMPS, Corollary 5.4], which is proved by using different methods. J. Schürmann informed us that such a result can also be obtained using the techniques of [Sch].

3 Milnor numbers

3.1 Lê's Milnor number

Lê D.T. [Lê3] proved that for a function f with an isolated singularity at $x_0 \in X$ (in the stratified sense) one has a Milnor fibration. He pointed out that, under certain conditions, the space X has "Milnor's property" in homology (which means that the reduced homology of the Milnor fiber of f is concentrated in dimension $\dim X - 1$). Then *the Milnor number* $\mu(f)$ is well defined as the rank of this homology group. By Lê's results [Lê3], Milnor's property is satisfied for instance if (X, x_0) is a complete intersection (not necessarily isolated!) or, more generally, if $\mathrm{rHd}\,(X, x_0) \geq \dim(X, x_0)$, where $\mathrm{rHd}\,(X, x_0)$ denotes the *rectified homology depth* of (X, x_0), see [Lê3] for its definition originating in Grothendieck's work.

To compare $\mu(f)$ with $\mathrm{Eu}_f(X, x_0)$ we use the general bouquet theorem for the Milnor fiber in its homological version. Let M_f and M_l denote the Milnor fiber of f and of a general function l. Let $f : (X, x_0) \to (\mathbb{C}, 0)$ be a function with stratified isolated singularity and let Λ be the set of stratified Morse points in some chosen Morsification of f (by convention $x_0 \notin \Lambda$). Then by [Ti, pp.228-229 and Bouquet Theorem] we have:

$$\tilde{H}_*(M_f) \simeq \tilde{H}_*(M_l) \oplus \oplus_{i \in \Lambda} H_{*-k_i+1}(C(F_i), F_i) \tag{1}$$

where, for $a_i \in \Lambda$, F_i denotes the complex link of the stratum to which a_i belongs, k_i is the dimension of this stratum and $C(F_i)$ denotes the cone over F_i.

In particular, if the germ (X, x_0) is a (singular) complete intersection (and more generally, if $\mathrm{rHd}\,(X, x_0) \geq \dim(X, x_0)$), then: $\mu(f) = \mu(l) + \sum_{a_i \in \Lambda} \mu_i$, where $\mu_i := \mathrm{rank}\, H_{\dim X - k_i}(C(F_i), F_i)$. This result shows that the Milnor number $\mu(f)$ gathers information from all stratified Morse points, whereas $\mathrm{Eu}_f(X, x_0)$ is, up to sign, the number $\alpha_q = \#\Lambda_0$, where Λ_0 denotes the set of Morse points occurring on X_{reg} (see Proposition 2.3 above). Notice that we have $\Lambda_0 \subset \Lambda$, $\mu(l) \geq 0$, $\mu_i = 1$ if $i \in \Lambda_0$ and $\mu_i \geq 0$ if $i \in \Lambda \setminus \Lambda_0$. We therefore get the general inequality, whenever the space X has Milnor's property (e.g. when (X, x_0) is a complete intersection, not necessarily with isolated singularities), and therefore the Milnor-Lê number of f is well defined:

$$\mu(f) \geq (-1)^{\dim X} \mathrm{Eu}_f(X, x_0). \tag{2}$$

In case (X, x_0) is a *complete intersection with isolated singularity*, ICIS for short, from the discussion following (1) we get the equality:

$$\mathrm{Eu}_f(X, x_0) = (-1)^{\dim X}[\mu(f) - \mu(l)]. \tag{3}$$

Still in case (X, x_0) is an ICIS case, the relation (3) also shows that the inequality (2) is strict, provided that x_0 is actually a singularity of X. Indeed, in this case we have $\mu(l) > 0$, which can be proved using Greuel's results for ICIS in [Gr], in particular his remarks on p. 264 after Proposition 5.7, *loc.cit.* Let us also point out that for certain complete intersections (X, x_0) with *nonisolated* singularity one might have the equality $\mu(l) = 0$ (for instance in case of equisingular hypersurfaces).

3.2 Another Milnor number

A different generalization of the Milnor number is due to V. Goryunov [Go], D. Mond and D. van Straten [MS]. This is originally defined for functions on curve singularities $X \subset \mathbb{C}^N$, and we refer to [MS, p.178] for the precise definition. This number is preserved under simultaneous deformations of both the space X and the function f. Thus, if the curve singularity (X, x_0) is a complete intersection with isolated singularity (ICIS), defined by some application $g\colon (\mathbb{C}^{p+1}, x_0) \to (\mathbb{C}^p, 0)$ on an open set in $\mathbb{C}^{p+1}$, and F is an extension of f to the ambient space, then $\mu_G(f)$ counts the number of critical points (with their multiplicities) of the restriction of F to a Milnor fiber of g, say $X_t = g^{-1}(t)$ for some regular value t of g. This makes sense for higher dimensional ICIS too and is equivalent to the fact that $\mu_G(f)$ is the Poincaré-Hopf index of the gradient of the restriction $F_{|X_t}$. In other words, this is saying that $\mu_G(f)$ is the *GSV-index* of the gradient vector field of f on X (the genuine gradient, not the conjugate of it as we have used up to now). We recall that the GSV-index of a vector field $\mathbf{v}$ on an ICIS (X, x_0), as defined in [GSV, SS], is the Poincaré-Hopf index of an extension of $\mathbf{v}$ to the Milnor fiber X_t.

We may notice that, for an ICIS, the invariant $\mu_G(f)$ equals the *virtual multiplicity* at x_0 of the function f on X introduced by Izawa and Suwa in [IS] and denoted $\tilde{m}(f; x_0)$. This multiplicity is by definition the localization at x_0 of the top Chern class of the virtual cotangent bundle $T^*(X)$ of X defined by the differential of f, which is non-zero on $X \setminus \{x_0\}$ by hypothesis. The virtual multiplicity has the advantage of being defined even if the singular set of X is non-isolated (we refer to [IS] for details).

The invariant $\mu_G(f)$, in case of an ICIS, also coincides with the index of the 1-form dg defined in [EG] and it is similar to the interpretation of the GSV index of vector fields given in [LSS] as a localization of the top Chern class of the virtual tangent bundle.

One can find the relation between $\mu_G(f)$ and $\mu(f)$ in case (X, x_0) is an ICIS, by using Greuel's [Gr, Lemma 5.3], as follows. Let $\mu(X, x_0)$ be the Milnor number of the ICIS (X, x_0) and let f be some function with isolated singularity on (X, x_0). Then:

$$\mu_G(f) = \mu(f) + \mu(X, x_0).$$

Using (3) we get:

$$\mathrm{Eu}_f(X, x_0) = (-1)^{\dim X}[\mu_G(f) - \mu_G(l)]. \tag{4}$$

These equalities completely clarify the relation between $\mathrm{Eu}_f(X, x_0)$, the GSV-index and the Milnor number of f, in terms of the Milnor number of the ICIS (X, x_0).

4 Further remarks

It is proved in [BMPS], using [BLS], that one has:

$$\mathrm{Eu}_f(X, x_0) = \sum_{i=0}^{q} [\chi(M(l, x_0) \cap W_i) - \chi(M(f, x_0) \cap W_i)] \cdot \mathrm{Eu}_X(W_i), \quad (5)$$

where $M(f, x_0)$ and $M(l, x_0)$ denote representatives of the Milnor fibers of f and of the generic linear function l, respectively. Combining this relation with Proposition 2.3 one gets:

$$\sum_{i=0}^{q} [\chi(M(l, x_0) \cap W_i) - \chi(M(f, x_0) \cap W_i)] \cdot \mathrm{Eu}_X(W_i) = (-1)^{\dim_{\mathbb{C}} X} \alpha_q.$$

Example 4.1. Let $X = \{x^2 - y^2 = 0\} \times \mathbb{C} \subset \mathbb{C}^3$ and f be the restriction to X of the function $(x, y, z) \mapsto x + 2y + z^2$. Take $x_0 := (0, 0, 0)$ and take as general linear function l the restriction to X of the projection $(x, y, z) \mapsto z$. Then X has two strata: $W_0 =$ the z-axis, $W_1 = X \setminus \{x = y = 0\}$. We compute $\mathrm{Eu}_f(X, x_0)$ from the relation (5).

First, $M(l, x_0) \cap W_0$ is one point and $M(f, x_0) \cap W_0$ is two points. Next, $M(l, x_0) \cap W_1$ is the disjoint union of two copies of $\mathbb{C}^*$ and $M(f, x_0) \cap W_1$ is the disjoint union of two copies of $\mathbb{C}^{**}$, where $\mathbb{C}^*$ is $\mathbb{C}$ minus a point and $\mathbb{C}^{**}$ is $\mathbb{C}$ minus two points. Then formula (5) gives: $\mathrm{Eu}_f(X, x_0) = (1 - 2) \cdot \mathrm{Eu}(X, x_0) + (0 - (-2)) \cdot 1$.

We have $\mathrm{Eu}(X, x_0) = \mathrm{Eu}(X \cap \{l = 0\}, x_0)$. Next $\mathrm{Eu}(X \cap \{l = 0\}, x_0)$ is just the Euler characteristic of the complex link of the slice $X \cap \{l = 0\} = \{x^2 - y^2 = 0\}$. This complex link is two points, so $\mathrm{Eu}(X \cap \{l = 0\}, x_0) = 2$. We therefore get $\mathrm{Eu}_f(X, x_0) = 0$.

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José Seade and **Alberto Verjovsky**
Instituto de Matemáticas
Unidad Cuernavaca
Universidad Nacional Autónoma de México
Apartado postal 273-3
C.P. 62210, Cuernavaca, Morelos
MÉXICO

E-mail: jseade@matem.unam.mx
E-mail: alberto@matcuer.unam.mx

Mihai Tibăr
Mathématiques, UMR 8524 CNRS
Université des Sciences et Technologies de Lille
59655 Villeneuve d'Ascq
FRANCE

E-mail: tibar@math.univ-lille1.fr